12/1/2021

Finding the computation from last lecture

(Flux of $\vec{v} = \langle z, y, x \rangle$ across sphere @ origin.)

We have $\iint_S \vec{v} \cdot d\vec{s} = \iint_{D_1} 0 \, d\vec{s} + \iint_{D_2} \sin^3(\phi) \sin^2(\theta) \, dA$

$$= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \sin^3(\phi) \sin^2(\theta) \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\pi} \sin^3(\phi) \frac{1}{2} \int_{\theta=0}^{2\pi} (1 - \cos(2\theta)) \, d\theta \, d\phi$$

$$= \frac{1}{2} \int_{\phi=0}^{\pi} \sin(\phi)(1 - \cos^2(\phi)) \left[\theta - \frac{1}{2} \sin(2\theta) \right]_{\theta=0}^{2\pi} d\phi$$

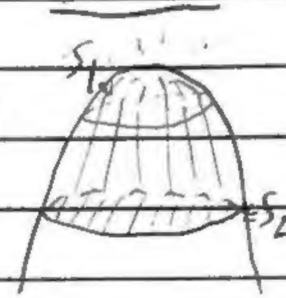
$$= \frac{1}{2}(2\pi - 0 - 0) \int_{\phi=0}^{\pi} -(1 - u^2) \, du$$

$$= -\pi \left[u - \frac{1}{3} u^3 \right]_{\phi=0}^{\pi} = -\pi \left[\cos(\phi) - \frac{1}{3} \cos^3(\phi) \right]_{\phi=0}^{\pi}$$

$$= -\pi \left(\left(-1 + \frac{1}{3}\right) - \left(1 - \frac{1}{3}\right) \right) = \frac{4\pi}{3} \quad \blacksquare$$

Ex: Compute the flux of $\vec{F} = \langle y, x, z \rangle$ on boundary of solid enclosed by paraboloid $z = 1 - x^2 - y^2$ and plane $z = 0$.

Picture



Sol: Our computation breaks up over the two pieces in our picture. (i.e. $S = S_1 \cup S_2$)

Parameterizations:

S_1: $\vec{S}(u,v) = \langle u\cos(v), u\sin(v), 1-u^2 \rangle$

(r,θ) $\quad D = [0,1] \times [0, 2\pi]$

$$\vec{F}(\vec{S}(u,v)) = \langle u\sin(v), u\cos(v), 1-u^2 \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{S}(u,v)) \cdot (\vec{S}_u \times \vec{S}_v)\, dA$$

$$\vec{S}_u = \langle \cos(v), \sin(v), -2u \rangle$$

$$\vec{S}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$$

$$\vec{S}_u \times \vec{S}_v = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(v) & \sin(v) & -2u \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix}$$

$\downarrow$

$= \langle 2u^2\cos(v), -(-2u^2\sin(v)), u\cos^2(v) + u\sin^2(v)\rangle$

$= u\langle 2u\cos(v), 2u\sin(v), 1\rangle = \vec{S}_u \times \vec{S}_v$

* Check orientation! *

(Check $u = \frac{1}{2}$, $v=0$, we see outward orientation)

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \langle u\sin(v), u\cos(v), 1-u^2\rangle \cdot u\langle 2u\cos(v), 2u\sin(v), 1\rangle\, dA$$

$$= \iint_D u\,(2u^2\sin(v)\cos(v) + 2u^2\cos(v)\sin(v) + (1-u^2))\, dA$$

$$= \int_{u=0}^{1} u\int_{v=0}^{2\pi} (4u^2\cos(v)\sin(v) + (1-u^2))\, dv\, du$$

$\int \cos(v)\sin(v)\, dv = \frac{1}{2}\sin^2(v) + C$

$$= \int_{u=0}^{1} u\left[2u^2\sin^2(v) + (1-u^2)v\right]_{v=0}^{2\pi} du$$

$$= \int_{u=0}^{1} u\,(0 + (1-u^2)(2\pi - 0))\, du$$

$$= 2\pi\int_{u=0}^{1}(u-u^3)\,du = 2\pi\left[\frac{1}{2}u^2 - \frac{1}{4}u^4\right]_{u=0}^{1}$$

$$= 2\pi\left(\frac{1}{2} - \frac{1}{4} - 0\right) = \pi\left(1 - \frac{1}{2}\right) = \frac{\pi}{2}$$

$\downarrow$

$\downarrow$

Now we work on S_2:

$S_2 \to \vec{r}(u,v) = \langle u\cos(v), u\sin(v), 0 \rangle$
on $D_2 = [0,1] \times [0, 2\pi]$

$\vec{F}(\vec{r}(u,v)) = \langle u\sin(v), u\cos(v), 0 \rangle$

$\vec{r}_u = \langle \cos(v), \sin(v), 0 \rangle$

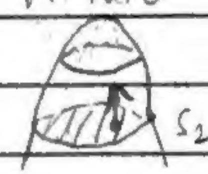
$\vec{r}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$\therefore \vec{r}_u \times \vec{r}_v = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix}$

$= \langle 0, 0, u\cos^2(v) + u\sin^2(v) \rangle$
$= u\langle 0, 0, 1 \rangle$

(Notice that this orientation is inward! So we need to use $-\vec{r}_u \times \vec{r}_v$ instead!) $\to$

Picture



$\therefore \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{D_2} \vec{F}(\vec{r}(u,v)) \cdot -(\vec{r}_u \times r_v)\, dA$

$= \iint_{D_2} \langle u\sin(v), u\cos(v), 0 \rangle \cdot -u\langle 0, 0, 1 \rangle \, dA$

$= \iint_{D_2} -u(0 + 0 + 0)\, dA = \iint_{D_2} 0\, dA = 0$

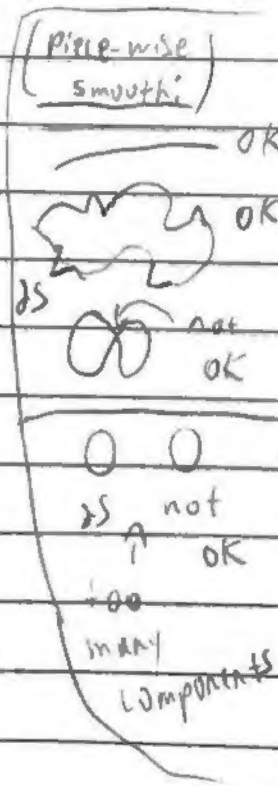
$\downarrow$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} = \frac{\pi}{2} + 0$$

$$= \frac{\pi}{2} \quad \blacksquare$$

§16.8: Stokes's Theorem

Idea: Want a version of Green's Theorem which does NOT require the surface to sit flat in $z=0$ plane.
(↳ Look back at the Green's Theorem notes...)



(Stokes's Theorem): Let S be a surface in $\mathbb{R}^3$ which is piecewise-smooth and with ∂S a piecewise-smooth closed curve with one component. If $\vec{F}$ is a v.f. on $\mathbb{R}^3$ w/ cts partial derivatives on S, then

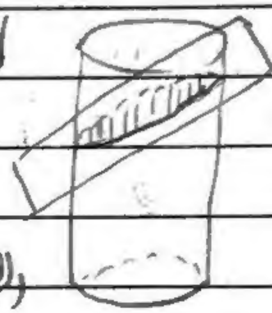
$$\int_{\partial S} \vec{F}\cdot dr = \iint_S \mathrm{curl}(\vec{F})\cdot d\vec{S}$$

NB: We'll take this as a black-box...
* (Can just know it as the formula above) *

Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and cylinder $x^2 + y^2 = 1$. (oriented counterclockwise from above)

Sol: We want to use Stokes's theorem, so we need $C = \partial S$ for surface S.

picture



Let's use the surface parameterized by:

$\vec{S}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), 2 - r\sin\theta \rangle$
on $D = [0,1] \times [0, 2\pi]$

Now by Stokes's Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

$$= \iint_D \text{curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta)\, dA$$

$$\text{curl}(\vec{F}) = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & z^2 \end{vmatrix}$$

↺

$$= \left\langle \frac{\partial}{\partial y}[z^2] - \frac{\partial}{\partial z}[x], -\left(\frac{\partial}{\partial x}[z^2] - \frac{\partial}{\partial z}[y^2]\right), \frac{\partial}{\partial x}[x] - \frac{\partial}{\partial y}[y^2] \right\rangle$$

$$= \langle 0, 0, 1 - 2y \rangle$$

$$\therefore \text{curl}(\vec{F})(\vec{S}(r,\theta)) = \langle 0, 0, 1 - 2r\sin(\theta) \rangle$$

$$\vec{S}_r = \langle \cos(\theta), \sin(\theta), -\sin(\theta) \rangle$$

$$\vec{S}_\theta = \langle -r\sin(\theta), r\cos(\theta), -r\cos(\theta) \rangle$$

$$\vec{S}_r \times \vec{S}_\theta = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -\sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) & -r\cos(\theta) \end{vmatrix}$$

$$= \langle -r\sin(\theta)\cos(\theta) + r\sin(\theta)\cos(\theta), -(-r\cos^2(\theta) - r\sin^2(\theta)), r\cos^2(\theta) + r\sin^2(\theta) \rangle$$

$$= \langle 0, r, r \rangle$$ orientation matches!

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_D \langle 0, 0, 1 - 2r\sin(\theta) \rangle \cdot \langle 0, r, r \rangle \, dA$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r(1 - 2r\sin(\theta)) \, d\theta \, dr$$

$$= \int_{r=0}^{1} r \left[\theta + 2r\cos(\theta) \right]_{\theta=0}^{2\pi} dr$$

$$= (2\pi - 0)\int_{r=0}^{1} r\,dr = 2\pi\left(\frac{1}{2}\left[r^2\right]_{r=0}^{1}\right)$$

$$= \pi(1-0) = \pi \quad \blacksquare$$

(Exercise: Verify this result directly (i.e. compute $\int_C \vec{F}\cdot dr$ using Stokes's Theorem))